University of Ottawa
Department of Mathematics and Statistics

MAT 1341 B: Introduction to Linear Algebra

Instructor: Prasad Senesi

Assignment 6
Due October 30 at 17:30 in the DGD (SITE B0138)

Family (Last) Name: ____________________

First Name: ____________________

Student Number: ____________________

Please read the following instructions.

- Enter your name and student number on this page, but enter only your name on the next page. The graded assignment will be returned to you without the first page.
- The table below is for the TA only. Do not write in the table.
- For privacy reasons, this page will not be returned to you. So **fill in your name on both pages** and your student number on this page only.

Question:	1	2	3	4	5	6	7	Total
Max points:	4	2	9	7	2	8	6	40
Score:								

Page 2

University of Ottawa
Department of Mathematics and Statistics
MAT 1341 B: Introduction to Linear Algebra
Instructor: Prasad Senesi

Assignment 6
Due October 30 at 17:30 in the DGD (SITE B0138)

Family (Last) Name: ______________________________

First Name: ______________________________

Please read the following instructions.

- Read each question carefully and answer all questions in the space provided. You may use the backs of pages if necessary, but indicate clearly when you have done this so that your answer is not overlooked.
- You must show your work for all questions or else no marks will be awarded.
- Please write legibly and argue logically. You must convince that TA that you know *why* your answer is correct.
- You must submit this assignment at the *beginning* of the DGD on October 30, at the latest. You may alternately submit the assignment to the professor in class, or at the Department of Mathematics and Statistics, 585 King Edward Ave, Room 103A between 13:00 and 17:00.

Question # 1: (1+1+1+1 = 4 points) Are the expressions compatible? Explain briefly why or why not.

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix}$

Solution: NO: 3×2 times 2×2 is not compatible

(b) $\begin{bmatrix} a & b & c \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ t \\ 9 \end{bmatrix}$

Solution: YES: 2×3 times 3×1 is compatible

(c) $\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Solution: NO: 2×3 times 3×1 gives 2×1 which can't be added to 3×1.

(d) M^3, where $M = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \end{bmatrix}$.

Solution: NO: 2×3 is not compatible with 2×3; only square matrices can be multiplied with themselves.

Question # 2: (2 points) Let $\mathbf{b}$ and $\mathbf{x}$ be column vectors (i.e., matrices with exactly one column), and A be an $m \times n$ matrix. Assume that their sizes are such that $A\mathbf{x} = \mathbf{b}$ is a valid equation. What are the sizes of $A, \mathbf{b}, \mathbf{x}$ such that $\mathbf{b}^T\mathbf{x}$ is guaranteed to exist? Explain why your answer is correct!

Solution: If A is $m \times n$ then $\mathbf{x}$ must have n rows for the product to be defined. Then $A\mathbf{x}$ will be a $m \times 1$ matrix. Then for the equation to make sense, $\mathbf{b}$ must also be a $m \times 1$ matrix.

For $\mathbf{b}^T\mathbf{x}$ to exist, $\mathbf{b}^T$ must have the same number of columns as $\mathbf{x}$ has rows. So $\mathbf{b}$ must have the same number of rows as $\mathbf{x}$ has rows. So $m = n$.

So $\mathbf{b}^T\mathbf{x}$ will exist if and only if A is square.

Question # 3: (3+3+3 = 9) Evaluate the given expressions.

(a) $\begin{bmatrix} 0 & 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 0 \\ 1 \end{bmatrix}$

Solution:

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1+t \\ t \\ 2 \end{bmatrix}$

Solution:

(c) $A^2\mathbf{x}$ where $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Solution:

Question # 3: (3+3+1 = 7 points) Let $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $D = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$.

(a) Evaluate $(CD)^T$.

Solution:

(b) Evaluate $C^T D^T$.

Solution:

(c) Show that $(CD)^T \neq C^T D^T$.

Solution:

Question # 4: (2 points) Is $\begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}$ in $Null(B)$, where $B = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 3 & -1 \\ 2 & -1 & -4 & 2 \end{bmatrix}$?

Solution: We multiply to get

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 3 & -1 \\ 2 & -1 & -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

No, the vector is not in $null(A)$.

Question # 5: (6+2 = 8 points) Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 3 & 3 & 2 & 1 \\ 0 & 1 & 2 & 6 \\ -1 & 0 & 0 & 7 \end{bmatrix}$.

(a) Describe the space $Col(A)$.
That is, give explicit conditions on a, b, c, d that exactly determine when $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is in $Col(A)$.

Solution: First we append the column-vector $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ to A and row-reduce.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & a \\ 0 & 0 & -1 & 1 & -3a+b \\ 0 & 1 & 2 & 6 & c \\ 0 & 1 & 1 & 7 & a+d \end{array}\right] \quad R_2 \to R_2 - 3R_1, \quad R_4 \to R_4 + R_1$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & a \\ 0 & 1 & 2 & 6 & c \\ 0 & 0 & -1 & 1 & -3a+b \\ 0 & 1 & 1 & 7 & a+d \end{array}\right] \quad R_3 \rightleftharpoons R_2$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & a \\ 0 & 1 & 2 & 6 & c \\ 0 & 0 & -1 & 1 & -3a+b \\ 0 & 0 & -1 & 1 & a-c+d \end{array}\right] \quad R_4 \to R_4 - R_2$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & a \\ 0 & 1 & 2 & 6 & c \\ 0 & 0 & 1 & -1 & 3a-b \\ 0 & 0 & -1 & 1 & a-c+d \end{array}\right] \quad R_3 \to -R_3$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & a \\ 0 & 1 & 2 & 6 & c \\ 0 & 0 & 1 & -1 & 3a-b \\ 0 & 0 & 0 & 0 & 4a-b-c+d \end{array}\right] \quad R_4 \to R_4 + R_3$$

Now we see that there will be a solution exactly when $4a - b - c + d = 0$. So

$$Col(A) = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \mid 4a - b - c + d = 0 \right\}$$

(b) Is $\begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix}$ in $Col(A)$?

Solution: We *could* append this vector to A and row-reduce, and see if we get a solution…But it's much quicker to use what we just computed: just check the condition:

$$4(1) - (-2) - (3) + (-4) = -1 \neq 0$$

So no, the vector is not in $Col(A)$.

Question # 6: (6 points) Reduce the following matrix to row-echelon form. Using this form, give a basis for its nullspace.

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 \\ 0 & 3 & 3 & 0 & 6 \\ 3 & 4 & 7 & 0 & 5 \end{bmatrix}$$

Solution: First we reduce it to row-echelon form.

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 \\ 0 & 3 & 3 & 0 & 6 \\ 3 & 4 & 7 & 0 & 5 \end{bmatrix} \xrightarrow[R_4 \to R_4 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & -2 & -1 & 1 & 0 \\ 0 & 5 & 5 & -2 & 0 \\ 0 & 3 & 3 & 0 & 6 \\ 0 & 10 & 10 & -3 & 5 \end{bmatrix}$$

$$\xrightarrow[R_2 \to \frac{1}{5}R_2]{} \begin{bmatrix} 1 & -2 & -1 & 1 & 0 \\ 0 & 1 & 1 & -\frac{2}{5} & 0 \\ 0 & 3 & 3 & 0 & 6 \\ 0 & 10 & 10 & -3 & 5 \end{bmatrix} \xrightarrow[R_4 \to R_4 - 10R_2]{R_3 \to R_3 - 3R_2} \begin{bmatrix} 1 & -2 & -1 & 1 & 0 \\ 0 & 1 & 1 & -\frac{2}{5} & 0 \\ 0 & 0 & 0 & \frac{6}{5} & 6 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow[R_3 \to \frac{5}{6}R_3]{} \begin{bmatrix} 1 & -2 & -1 & 1 & 0 \\ 0 & 1 & 1 & -\frac{2}{5} & 0 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix} \xrightarrow[R_4 \to R_4 - R_3]{} \begin{bmatrix} 1 & -2 & -1 & 1 & 0 \\ 0 & 1 & 1 & -\frac{2}{5} & 0 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R_1 \to R_1 - R_3]{R_2 \to R_2 + \frac{2}{5}R_3} \begin{bmatrix} 1 & -2 & -1 & 0 & -5 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 \to R_1 + 2R_2]{} \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we read off the solution. There are two parameter, $x_3 = s$ and $x_5 = t$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s + t \\ -s - 2t \\ s \\ -5t \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 0 \\ -5 \\ 1 \end{bmatrix}$$

So a basis for $null(A)$ is

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ -5 \\ 1 \end{bmatrix} \right\}$$